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A general solution for dynamic response of axially loaded non-uniform Timoshenko beams

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Abstract

A dynamic investigation method for the analysis of Timoshenko beams which takes into account the shearing deformation and the rotating inertia is proposed.

The solution of the problem is obtained through the iterative variational Rayleigh–Ritz method and assuming as test functions an appropriate class of orthogonal polynomials which respect the essential conditions only. The procedure, applied to tapered beams for which a closed form solution is not known, is an alternative approach to the usual FEM methodologies used in literature. The small number of Lagrangean parameters needed for the analysis allows the use of strict symbolic calculation programs obtaining an high numerical accuracy with a relative short computer time. The work ends with the analysis of a few numerical examples and the results are compared with the ones obtained from other authors mentioned in bibliography.

Keywords: Dynamic; Stability; Timoshenko beams; Axial-loads

1. Introduction

The dynamic behaviour of structural elements with both complex geometry, consistent shearing deformation and rotating inertia can be studied by means of the Euler–Bernoulli beam theory. As well known, an exact analysis can be carried out only in the case of constant section beam. If we want to take into account the section variation we can only perform approximate analysis mostly based on finite element methods (Downs, 1977). In particular To (1981), extended the theory developed by Przemieniecki (1968) for constant section beams, presenting an excellent work on the free-frequencies determination by using cubic test functions. Recently, Cleghorn and Tabarrok (1992) reformulated the problem obtaining the exact

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stiffness matrix of the beam element. Adopting a beam-like element with four degrees of freedom, Eisenberger (1994, 1995) obtained the first 10 exact free frequencies for simple supported tapered beams.

Adopting a semi-analytical procedure (Courant and Hilbert, 1953), Gutierrez et al. (1991) obtained the fundamental frequencies of tapered beams with a mass to the extreme reducing the eigenvalue problem to a minimum problem. The Rayleigh-like approach followed in the paper is alternative to the usual FEM models and is based on the optimization of an exponential Schmidt (1982) parameter. Another alternative to the numerous variational approaches is given by Auciello (1993, 2000) in which, applying the force method the structure is discretized in rigid elements connected by elastic skates. The solution is obtained exploiting the classical theorems for Lagrangian approach (LA) in which the relative vertical shiftings are assumed as freedom parameters. As regards axially loaded beams many contributions from different Authors can be found in literature. Among the others an exact solution referred to constant section beams has been given by Abramovich (1992). For tapered beams with axial loads it is worthwhile to cite the recent work of Leung and Zhou (1995) where the dynamic stiffness matrix is widely used for the determination of the free frequencies.

A detailed analysis is carried out by Esmailzadeh and Ohadi (2000) which consider the equation of motion for two different positions of the axial load and obtain the free frequencies by means of the Frobenius method.

This paper presents a general technique for the evaluation of the free frequencies based on the Rayleigh–Ritz method. The energy functional takes into account both the shear deformation and the rotatory inertia. Displacements and rotations are given through a suitable choice of orthogonal polinomials which have already supplied excellent results both for slab elements (Bhat, 1985) and Euler–Bernoulli beams (Auciello, 2001). The numerical solutions of the examples have been obtained by means of Mathematica (Wolfram, 1991).

2. Problem formulation

Let us consider a variable-section beam of length L made of an homogeneous and isotropic material. We also defined a cartesian frame $\{0, x_i, i = 1, 2, 3\}$ where x_1 coincides with axis of the beam and x_2 and x_3 are the principal axis of the cross section. Obviously, if loads have the direction of x_2 we can write for the displacements

$$\begin{aligned} u_1 &= -x_2\phi(x_1, t) \\ u_2 &= u_2(x_1, t), \\ u_3 &= 0 \end{aligned} \tag{1}$$

where $\phi(x_1, t)$ is the cross-section rotation.

As well known, in the Timoshenko model the cross-section rotation no longer coincides with the tangent to the section and consequently we write (Fig. 1):

$$\frac{\partial u_2}{\partial x_1} = \phi + \gamma. \tag{2}$$

By $\gamma = \frac{Q}{GA}$ we denoted the average shear deformation. The non-zero components of the deformation tensor are:

$$\varepsilon_{11} = (-x_2\phi') \quad \varepsilon_{12} = \frac{1}{2}(u_2' - \phi). \tag{3a, b}$$

where by the apex we denoted the derivative with respect to x_1.

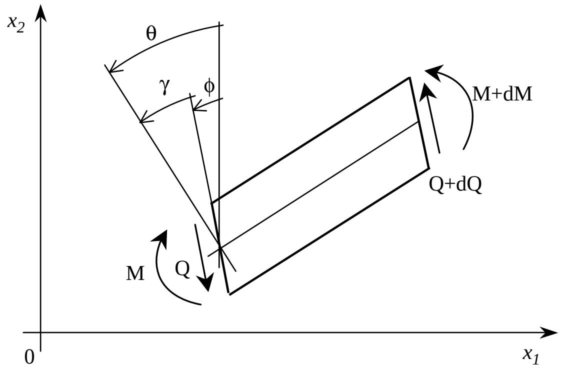

Fig. 1. Timoshenko beam element: kinematics of deformation.

Consequently the elastic energy of the structure is

$$U = \frac{1}{2}\int_0^L \int_A \boldsymbol{\sigma}^{\mathrm{T}}\boldsymbol{\varepsilon}\,\mathrm{d}V, \tag{4}$$

where

$$\boldsymbol{\varepsilon} = \begin{bmatrix} -x_2\phi' & -\phi + u_2' \end{bmatrix}^{\mathrm{T}}. \tag{5}$$

Taking into account the constitutive relations (5) and (4) can be rewritten as

$$U = \frac{1}{2}\int_0^L \begin{bmatrix} \phi' \\ u_2' - \phi \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} EI & 0 \\ 0 & kGA \end{bmatrix} \begin{bmatrix} \phi' \\ u_2' - \phi \end{bmatrix} \mathrm{d}x_1, \tag{6}$$

where k is the shear factor of the sections (Cowper, 1966), and EI and kGA are the flexural and shear stiffness respectively.

The potential energy of the applied forces is given by the axial forces P and is a second order function of the deformation tensor:

$$V_P = \frac{P}{2}\int_0^L (u_2')^2\,\mathrm{d}x_1. \tag{7}$$

The kinetic energy can be written

$$T = \frac{1}{2}\int_0^L \int_A \rho(\dot{u}_1^2 + \dot{u}_2^2 + \dot{u}_3^2)\,\mathrm{d}V. \tag{8}$$

After the area integration (8) becomes:

$$T = \frac{1}{2}\int_0^L \begin{bmatrix} \dot{\phi} \\ \dot{u}_2 \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} \rho I & 0 \\ 0 & \rho A \end{bmatrix} \begin{bmatrix} \dot{\phi} \\ \dot{u}_2 \end{bmatrix} \mathrm{d}x_1. \tag{9}$$

If we are searching for solutions of the kind

$$\begin{aligned} u_2(x_1,t) &= u_2(x_1)\,\mathrm{e}^{\mathrm{i}\omega t}, \\ \phi(x_1,t) &= \phi(x_1)\,\mathrm{e}^{\mathrm{i}\omega t}, \end{aligned} \tag{10}$$

the kinetic energy can also written:

$$T = \frac{\omega^2}{2} \int_0^L \begin{bmatrix} u_2 \\ \phi \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} \rho A & 0 \\ 0 & \rho I \end{bmatrix} \begin{bmatrix} u_2 \\ \phi \end{bmatrix} \mathrm{d}x_1. \tag{11}$$

Finally we obtain the functional T:

$$(U + V_p) = T. \tag{12}$$

3. Approximate solution

Obviously the solution is affected by the choice of the functions which represent kinematics. According to the usual Rayleigh–Ritz (R–R) approximation, u_2 and ϕ are linear combinations of functions which respect the essential conditions and take the form:

$$u_2(x_1) = \sum_{k=1}^{n} a_k \varphi_k = \mathbf{\Phi}^{\mathrm{T}} \mathbf{q}_1, \tag{13}$$

$$\phi(x_1) = \sum_{k=1}^{n} b_k \psi_k = \mathbf{\Psi}^{\mathrm{T}} \mathbf{q}_2, \tag{14}$$

where

$$\mathbf{q}_1 = [a_1 \quad a_2 \quad \cdots \quad a_n]^{\mathrm{T}}, \quad \mathbf{q}_2 = [b_1 \quad b_2 \quad \cdots \quad b_n]^{\mathrm{T}} \tag{15}$$

are the generalized Lagrangian coordinates. Each component of the following vectors:

$$\mathbf{\Phi} = [\varphi_1 \quad \varphi_2 \quad \varphi_3 \quad \cdots \quad \varphi_n]^{\mathrm{T}}, \quad \mathbf{\Psi} = [\psi_1 \quad \psi_2 \quad \psi_3 \quad \cdots \quad \psi_n]^{\mathrm{T}}, \tag{16}$$

represents a well specified type of function defined in the interval $[0, L]$.

Substituting (13) and (14) into (6) we get the expression of the elastic energy:

$$U = \frac{1}{2} \int_0^L [EI(b_k \psi_k)'^2 + kGA(a_k \phi_k' - b_k \psi_k)^2] \, \mathrm{d}x_1. \tag{17}$$

The elastic energy in (17) can also be written in matrical form as

$$U = \frac{1}{2} \mathbf{q}^{\mathrm{T}} \left\{ E \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{B}_2 \end{bmatrix} + kG \begin{bmatrix} \mathbf{B}_1 & -\mathbf{B}_3^{\mathrm{T}} \\ -\mathbf{B}_3 & \mathbf{B}_0 \end{bmatrix} \right\} \mathbf{q} = \frac{1}{2} \mathbf{q} \mathbf{K}_U \mathbf{q}, \tag{19}$$

where

$$\mathbf{q} = [\mathbf{q}_1 \quad \mathbf{q}_2]^{\mathrm{T}}, \tag{18}$$

and

$$\mathbf{B}_1 = \int_0^L A \mathbf{\Psi} \mathbf{\Psi}^{\mathrm{T}} \, \mathrm{d}x_1, \quad \mathbf{B}_0 = \int_0^L A \mathbf{\Phi}' \mathbf{\Phi}'^{\mathrm{T}} \, \mathrm{d}x_1, \tag{20, 21}$$

$$\mathbf{B}_2 = \int_0^L I \mathbf{\Psi}' \mathbf{\Psi}'^{\mathrm{T}} \, \mathrm{d}x_1, \quad \mathbf{B}_3 = \int_0^L A \mathbf{\Psi} \mathbf{\Phi}'^{\mathrm{T}} \, \mathrm{d}x_1. \tag{22, 23}$$

The stiffness matrix $\mathbf{K}_U$ contains both the flexural and shear deformation energy.

In the same fashion the potential energy of the applied loads can be written as

$$V_P = \frac{P}{2}\int_0^L (a_k\,\phi_k)'^2\,\mathrm{d}x_1 = \frac{P}{2}\mathbf{q}^{\mathrm{T}}\begin{bmatrix}\mathbf{B}_P & \mathbf{0}\\ \mathbf{0} & \mathbf{0}\end{bmatrix}\mathbf{q}, \tag{24}$$

where

$$\mathbf{B}_P = \int_0^L \mathbf{\Phi}'\mathbf{\Phi}'^{\mathrm{T}}\,\mathrm{d}x_1. \tag{25}$$

Substituting (13) into (11) we get for the kinetic energy T the expression

$$T = \frac{\omega^2}{2}\int_0^L \rho[A(a_k\phi_k)^2 + I(b_k\psi_k)^2]\,\mathrm{d}x_1, \tag{24}$$

In matricial form

$$T = \frac{\omega^2}{2}\mathbf{q}^{\mathrm{T}}\rho\begin{bmatrix}\mathbf{B}_4 & \mathbf{0}\\ \mathbf{0} & \mathbf{B}_5\end{bmatrix}\mathbf{q} = \frac{\omega^2}{2}\mathbf{q}^{\mathrm{T}}\mathbf{M}\mathbf{q}. \tag{25}$$

where

$$\mathbf{B}_4 = \int_0^L A\mathbf{\Phi}\mathbf{\Phi}^{\mathrm{T}}\,\mathrm{d}x_1, \quad \mathbf{B}_5 = \int_0^L I\mathbf{\Psi}\mathbf{\Psi}^{\mathrm{T}}\,\mathrm{d}x_1. \tag{26}$$

Obviously the matrix $\mathbf{M}$ is symmetric and positive definite, and is comprehensive of the rotatory inertia.

At last, functional in (12) is written

$$\Pi[\mathbf{q}] = \frac{1}{2}\mathbf{q}^{\mathrm{T}}(\mathbf{K}_U + \mathbf{K}_P)\mathbf{q} - \frac{\omega^2}{2}\mathbf{q}^{\mathrm{T}}\mathbf{M}\mathbf{q} = \frac{1}{2}\mathbf{q}^{\mathrm{T}}(\mathbf{K} - \omega^2\mathbf{M})\mathbf{q}, \tag{27}$$

where $\mathbf{K}$ is the stiffness matrix. It is worth to note that, for $P < 0$, $\mathbf{K}_P$ *is negative* definite so that the overall effect is a reduction of the eigenvalues of $\mathbf{K}$. By the stationary condition of the functional in (27) we get:

$$\begin{aligned}\delta\Pi &= \frac{1}{2}\partial\mathbf{q}^{\mathrm{T}}(\mathbf{K} - \omega^2\mathbf{M})\mathbf{q} + \frac{1}{2}\mathbf{q}^{\mathrm{T}}(\mathbf{K} - \omega^2\mathbf{M})\partial\mathbf{q} = \frac{1}{2}\partial\mathbf{q}^{\mathrm{T}}(\mathbf{K} - \omega^2\mathbf{M})\mathbf{q} + \frac{1}{2}[(\mathbf{K} - \omega^2\mathbf{M})\partial\mathbf{q}]^{\mathrm{T}}\mathbf{q}\\ &= \frac{1}{2}\partial\mathbf{q}^{\mathrm{T}}(\mathbf{K} - \omega^2\mathbf{M})\mathbf{q} + \frac{1}{2}\partial\mathbf{q}^{\mathrm{T}}(\mathbf{K} - \omega^2\mathbf{M})\mathbf{q} = \partial\mathbf{q}^{\mathrm{T}}(\mathbf{K} - \omega^2\mathbf{M})\mathbf{q} = 0,\end{aligned} \tag{28}$$

From the variational equality (28) we get the following homogeneous system in the unknown $\mathbf{q}$

$$(\mathbf{K} - \omega^2\mathbf{M})\mathbf{q} = \mathbf{0}. \tag{29}$$

The frequency equation is given by the well known equation:

$$\det(\mathbf{K} - \omega^2\mathbf{M}) = 0. \tag{30}$$

4. Test functions

A general procedure which makes use of vectors $\mathbf{\Phi}$ and $\mathbf{\Psi}$ respecting only the essential conditions is presented. In particular orthogonal polynomial functions are assumed starting from a first polynomial p_1 of grade equal to the number of the essential conditions. Obviously every $p_i \in L_2[0, L]$ is square integrable; i.e:

$$\int_0^L p_i\,\mathrm{d}x_1 < \infty, \quad \langle p_i, p_i\rangle = \int_0^L p_i p_i\,\mathrm{d}x_1 < \infty. \tag{31}$$

The choice of these polynomials has proved to be very effective. The functions p_i are obtained by means of the Gram–Schmidt iterative procedure presented by Bhat (1985) and already utilized for Euler–Bernoulli beams in Auciello (2001). In this case we first set all the φ_k related to the transversal displacements and subsequently the ψ_k related to the shear deformation. We start with the following polynomial:

$$\varphi_1(x_1) = c_j x_1^j, \quad j = 0, m \tag{32}$$

where m is equal to the number of the essential conditions. Obviously we have to satisfy the normality condition too:

$$\langle \varphi_1, \varphi_1 \rangle = 1. \tag{33}$$

The other polynomials φ_k with $k > r$ are obtained through the sequence:

$$\varphi_k = (x - D_k)\varphi_{k-1} - C_k \varphi_{k-2}, \tag{34}$$

where

$$D_k = \frac{(x\varphi_{k-1}, \varphi_{k-1})}{(\varphi_{k-1}, \varphi_{k-1})}, \quad C_k = \frac{(x\varphi_{k-1}, \varphi_{k-2})}{(\varphi_{k-2}, \varphi_{k-2})}. \tag{35}$$

The remaining functions ψ_k are obtained from relative rotation conditions starting from a polynomial of an order lower than the one chosen for the transversal displacements and repeating the sequence ((32)–(35)) after the substitution of φ_k with ψ_k. In other words the polynomial function related to the transversal displacements presents an order greater than the one related to rotation functions. This particular choice is made in order to avoid the overestimation of the rate of elastic energy due to shear respect to the rate due to bending. Otherwise we can't simulate the Euler–Bernoulli beam. In fact rewriting Eq. (6):

$$U = \frac{1}{2}\int_0^L [EI(\phi')^2 + kGA(u_2' - \phi)^2]\,\mathrm{d}x_1 = \frac{1}{2}\int_0^L EA\left[\frac{I}{A}(\phi')^2 + k\frac{G}{E}(u_2' - \phi)\right]\mathrm{d}x_1 = U_b + U_s, \tag{36}$$

we note that for $u_2' \to \phi$ we get the deformation energy in the Euler–Bernoulli theory:

$$U_e = \frac{1}{2}\int_0^L EI(u_2'')^2\,\mathrm{d}x_1. \tag{37}$$

Consequently if we use polynomials of the same order for both displacements and rotations we get that in (36) the ratio I/A goes to zero faster than the difference $(u_2' - \phi)$. Then the energy rate U_s, which does not depend from I/A, overcomes the rate U_b and the resulting elastic energy is erroneously much bigger than the value in (37). This is a well-known numerical phenomenon defined in literature as shear-locking. It can take place in the finite elements procedures when shear deformation and rotatory inertia are taken into account; Carpenter et al. (1986). In other words, because the problem is strongly influenced by the choice of the test functions, we could risk to increase the degrees of freedom of the structure without obtaining an improvement of the solution.

Using the symbolic program *Mathematica* the sequence ((32)–(35)) have been implemented in a numerical procedure which allows the calculation of the test functions satisfying the m essential conditions. In the end assuming polynomial functions respectively of order $m-1$ for u_2 and of order $m-1$ for ϕ we arrive to a *consistent interpolation element* type (Reddy, 1997).

Obviously, in order to avoid completely the shear locking higher order methods should be preferred Reddy (1997).

5. Results and discussion

In what follows, some problems, related to different boundary conditions, are discussed.

The numerical examples shown deal with beams made of homogeneous and isotropic material with a ratio E/G equal to 2.6 and a shear correction factor k equal to 5/6. Although the method presented can be applied to more complex structure we assume a beam section with a constant base B and a linearly variable height H. In these conditions the geometrical parameters of the cross-section are given as functions of α:

$$A(x_1) = A_0(1 + \alpha x_1), \quad I(x_1) = I_0(1 + \alpha x_1)^3. \tag{38, 39}$$

By A_0 and I_0, we have denoted respectively the area and the inertia at $x_1 = 0$.

In the following two examples are given.

5.1. Cantilever beam with tip mass

Let us consider the case of the cantilever beam with a tip mass and rotatory inertia (Fig. 2a) already reported by Leung and Zhou (1995).

Obviously we have to add in (11) the rate of kinetic energy due to the rotatory inertia:

$$T_M = \frac{\omega^2}{2}[M\dot{u}_2^2(L) + J_M\dot{\phi}^2(L)]. \tag{40}$$

From (13 and 14) we get

$$T_M = \frac{1}{2}\omega^2\mathbf{q}^{\mathrm{T}}\begin{bmatrix}\mathbf{B}_6 & 0\\ 0 & \mathbf{B}_7\end{bmatrix}\mathbf{q} = \frac{\omega^2}{2}\mathbf{q}^{\mathrm{T}}\widetilde{\mathbf{M}}\mathbf{q}, \tag{41}$$

where

$$\mathbf{B}_6 = M\mathbf{\Phi}_L\mathbf{\Phi}_L^{\mathrm{T}}, \quad \mathbf{B}_7 = J_M\mathbf{\Psi}_L\mathbf{\Psi}_L^{\mathrm{T}} \tag{42}$$

Obviously we have to add (41) into (26).

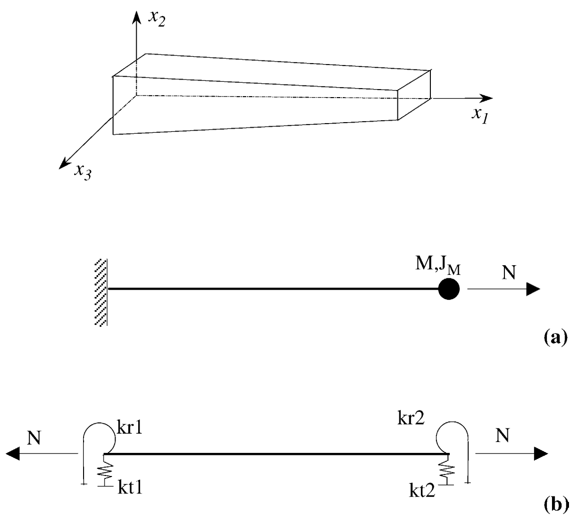

Fig. 2. Beam of non-uniform thickness under compressive axial loads. (a) Cantilever beam with tip mass. (b) A general elastically end restrained-beam.

In the following we will make use of these two non-dimensional parameters which are frequently used in literature:

$$r = \sqrt{\frac{I_0}{A_0}}, \quad P_r = -\frac{NL^2}{EI_0} \tag{43, 44}$$

Free frequencies are expressed through the parameters

$$\Omega_i^2 = \omega_i L^2 \sqrt{\frac{\rho A_0}{EI_0}}. \tag{45}$$

As first example the case of the cantilever without axial loads with tapering parameter $\alpha = -0.2$ is reported. This case has been analysed by Leung and Zhou (1995) using the dynamic stiffness method (DSM), by Rossi et al. (1990) using a finite element approach (FEM) and by Auciello (2000) who used a LA.

Table 1
Cantilever tapered beam with a tip mass; first three frequency coefficients for various values of n

$\alpha = -0.2$	$J_M = 0$		n							Rossi et al. (1990)	Error (%)	Error (%)
r	μ	Ω	2	3	4	5	6	7	8		$n8$/Rossi et al. (1990)	$n5/n8$
Clamped-free												
0.02	0	Ω_1	4.156	3.598	3.597	3.596	3.595	3.595	3.595	3.59	0.139	0.028
		Ω_2		27.288	20.317	20.273	20.18	20.18	20.18	20.17	0.05	0.461
		Ω_3				54.859	54.355	53.488	53.488	53.48	0.015	**2.522**
	0.4	Ω_1	2.348	2.146	2.144	2.144	2.144	2.144	2.144	2.14	0.187	
		Ω_2		18.754	15.646	15.536	15.526	15.525	15.525	15.52	0.032	0.071
		Ω_3				46.133	45.243	45.05	45.026	45.03	0.009	2.447
	1	Ω_1	1.652	1.527	1.525	1.525	1.524	1.524	1.524	1.52	0.262	
		Ω_2		17.562	14.855	14.734	14.728	14.726	14.726	14.72	0.041	0.054
		Ω_3				45.228	44.293	44.133	44.113	44.12	0.016	2.517
0.04	0	Ω_1	4.26	3.561	3.559	3.558	3.558	3.558	3.558	3.56	0.056	
		Ω_2		23.64	19.128	19.066	19.019	19.018	19.018	19.01	0.042	0.252
		Ω_3				48.302	47.796	47.398	47.396	47.43	0.072	1.896
	0.4	Ω_1	2.3	2.131	2.129	2.129	2.129	2.129	2.129	2.13	0.047	
		Ω_2		16.993	14.895	14.826	14.82	14.819	14.819	14.82	0.007	0.047
		Ω_3				41.082	40.612	40.504	40.495	40.52	0.062	1.445
	1	Ω_1	1.622	1.517	1.516	1.516	1.516	1.516	1.516	1.51	0.396	
		Ω_2		15.98	14.154	14.081	14.076	14.076	14.076	14.07	0.043	0.036
		Ω_3				40.279	39.798	39.707	39.7	39.72	0.05	1.455
0.08	0	Ω_1	3.703	3.423	3.422	3.422	3.422	3.422	3.422	3.42	0.058	
		Ω_2		17.665	15.899	15.854	15.84	15.84	15.84	15.84	0	0.088
		Ω_3				35.64	35.385	35.271	35.271	35.35	0.224	1.043
	0.4	Ω_1	2.177	2.075	2.074	2.074	2.074	2.074	2.074	2.07	0.193	
		Ω_2		13.661	12.787	12.761	12.758	12.758	12.758	12.76	0.016	0.024
		Ω_3				31.049	30.9037	30.865	30.863	30.92	0.185	0.602
	1	Ω_1	1.543	1.482	1.481	1.481	1.481	1.481	1.481	1.48	0.068	
		Ω_2		12.952	12.189	12.164	12.162	12.162	12.162	12.16	0.016	0.016
		Ω_3				30.445	30.303	30.271	30.269	30.32	0.168	0.581

Because of (39), the applied mass is a function of the following parameters:

$$\mu = M\left(1+\frac{\alpha}{2}\right)^{-1}, \quad J^* = \sqrt{\frac{J_M}{M}}. \tag{46}$$

In order to verify the convergence properties of the procedure, the first three frequencies, for different values of r and n, are reported in Table 1. By n we denoted the number of the degrees of freedom used in the discretization. In the last column the approximation for $n = 8$ is reported. It appears clearly that the differences increase for smaller values of r. This is due to the shear locking phenomenon which, although minimized, becomes measurable as $r \to 0$ (Table 1).

The results given by different authors are reported in Table 2. In particular, those given by Leung and Zhou (1995) and written in Italics are obtained using power series stopped to the 25th power and they are practically coinciding with the results of this paper. As well known, in the LA method we get to the solution from below so that the frequencies are lower than the exact ones. It is worth of note that for $n = 7$ the model has $2n = 14°$ of freedom and achieves a very good accuracy with substantially few degrees of freedom.

In Fig. 3, we have the variation of Ω_1 for changes in the axial load and for two different values of $\mu : \mu = 0$ (3a) and $\mu = 1$ (3b). It is worth to notice that as P_r grows the fundamental frequencies decay and go to zero as $P_r \to P_c$. For the same α, the applied mass causes a decrease of natural frequencies. This can be relevant in the seismic design where applied mass are introduced in order to change the natural period and avoid resonance problems.

5.2. The beam with elastic constraints

The boundary conditions are referred to the scheme in Fig. 2b. The kinetic energy is still given by (11). We just have to add in (6) the elastic energy of the constraints U_c:

$$U_c = \frac{1}{2}k_{r1}\varphi(0) + \frac{1}{2}k_{r2}\varphi(L) + \frac{1}{2}k_{t1}u_2^2(0) + \frac{1}{2}k_{t2}u_2^2(L). \tag{47}$$

Table 2
Cantilever tapered beam: comparison between the present method, the dynamic stiffness method, Leung and Zhou (1995), the finite element method (FEM), Rossi et al. (1990), and the Lagrangian approach (LA), Auciello (2000)

α	$J_M = 0$	Present (−0.2)			Leung and Zhou (1995)			−0.2 LA (Rossi et al., 1990)			−0.2, LA (Auciello, 2000)		
r	μ	Ω_1	Ω_2	Ω_3	Ω_1	Ω_2	Ω_3	Ω_1	Ω_2	Ω_3	Ω_1	Ω_2	Ω_3
Clamped-free													
0.02	0	3.587	20.18	53.488				3.59	20.17	53.48	3.584	19.984	52.445
	0.4	2.144	15.525	45.05				2.14	15.52	45.03	2.002	15.228	44.219
	1	1.524	14.726	44.133				1.52	14.72	44.12	1.402	14.542	43.474
0.04	0	3.558	19.018	47.398				3.56	19.01	47.43	3.552	18.855	46.693
	0.2	2.588	15.671	41.542	*2.59*	*15.67*	*41.53*						
	0.4	2.129	14.819	40.504				2.13	14.82	40.52	1.99	14.559	39.474
	0.6	1.85	14.435	40.081									
	1	1.516	14.076	39.707				1.51	14.07	39.72	1.393	13.921	39.328
0.08	0	3.422	15.84	35.271				3.42	15.84	35.35	3.415	15.744	34.96
	0.4	2.074	12.758	30.865				2.07	12.76	30.92	1.94	12.585	30.708
	1	1.481	12.162	30.271				1.48	12.16	30.32	1.362	12.074	30.237
0.1	0	3.33	14.289	30.711	*3.3*	*14.31*	*30.7*						
	0.2	2.461	12.268	27.726	*2.46*	*12.27*	*27.73*						
	0.6	1.773	11.418	26.793	*1.77*	*11.42*	*26.79*						

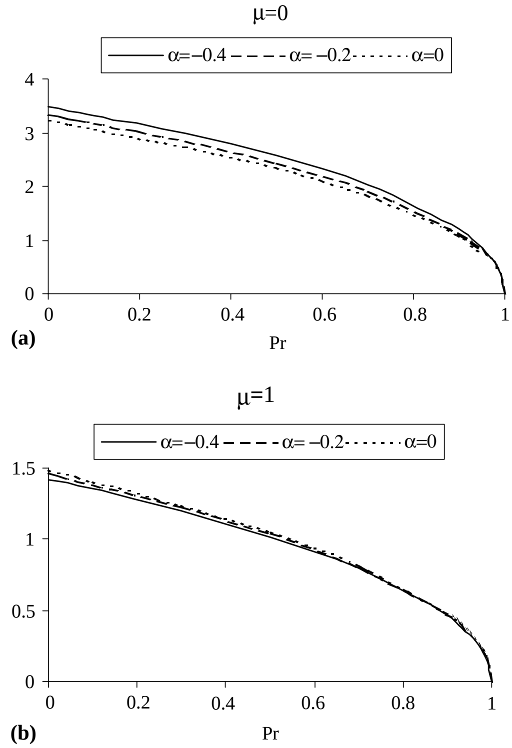

Fig. 3. (a) Fundamental frequency vs. axial-force parameter for cantilever beam with a tip mass: (a) $\mu = 0$ (b) $\mu = 1$.

Using (13) and (14), after some algebra we get to the expression

$$U_c = \frac{1}{2}\begin{bmatrix} \mathbf{B}_8 & \mathbf{0} \\ \mathbf{0} & \mathbf{B}_9 \end{bmatrix} \tag{48}$$

where

$$\mathbf{B}_8 = k_{t1}\mathbf{\Phi}(0)\mathbf{\Phi}^{\mathrm{T}}(0) + k_{t2}\mathbf{\Phi}(L)\mathbf{\Phi}^{\mathrm{T}}(L), \tag{49}$$

$$\mathbf{B}_9 = k_{r1}\mathbf{\Psi}(0)\mathbf{\Psi}^{\mathrm{T}}(0) + k_{r2}\mathbf{\Psi}(L)\mathbf{\Psi}^{\mathrm{T}}(L). \tag{50}$$

For numerical convenience it is better to introduce the following non-dimensional parameters:

$$K_{T1} = \frac{k_{t1}L^3}{EI(0)}, \quad K_{T2} = \frac{k_{t2}L^3}{EI(L)}, \quad K_{R1} = \frac{k_{r1}L}{EI(0)}, \quad K_{R2} = \frac{k_{r2}L}{EI(L)}. \tag{51}$$

The simply supported beam has been analysed for $K_{T1} \to \infty$, $K_{T2} \to \infty$ and $K_{R1} = 0$, $K_{R2} = 0$. In Fig. 4 the fundamental frequency parameter Ω_1 vs. axial load for different α are reported.

In Table 3 the first five free frequencies for the pinned–pinned beam and for different values of α are compared with the exact values given by Eisenberger (1995).

In order to obtain the upper frequencies the discretization parameter n has been chosen equal to 10 so that the stiffness matrix has dimension equal to 20.

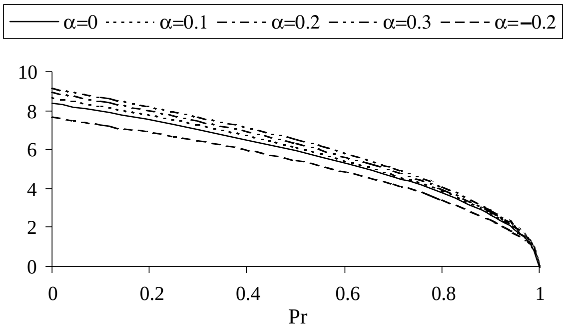

Fig. 4. Fundamental frequency vs. axial-force parameter for simply supported beam.

Table 3
Comparison of the first five non-dimensional frequencies for simply supported beam between present method and exact approach (Eisenberger, 1995)

α	−0.5		0		1	
$r = 0.0707$	Present	Eisenberger (1995)	Present	Eisenberger (1995)	Present	FEM (Rossi et al., 1990)
Pinned–pinned						
$\Omega 1$	6.765	6.754	9.023	9.023	11.901	11.896
$\Omega 2$	24.462	24.539	29.914	29.912	36.427	36.424
$\Omega 3$	47.371	47.302	55.201	55.207	63.004	62.849
$\Omega 4$	72.674	72.672	81.817	81.815	68.143	68.048
$\Omega 5$	99.231	99.039	108.856	108.695	89.984	89.696

In Table 4 the values of Ω_1 as function of r are reported and compared with the ones of other authors. The solution obtained by Gutierrez et al. (1991) using a finite element procedure proposed by Przemieniecki (1968) is almost coincident with the one presented in this paper. The same authors, Gutierrez et al. (1991) developed a different procedure starting from the Ritz approach and using an optimization factor proposed

Table 4
Comparison of the first non-dimensional frequency Ω_1, between present method and Lagrangian approach (LA) for simply supported beam

α	0				0.1				0.15			
r	Present	Exact	Ritz approach (Gutierrez et al., 1991)	LA (Auciello, 1993)	Present	FEM	Ritz approach (Gutierrez et al., 1991)	LA (Auciello, 1993)	Present	FEM	Ritz approach (Gutierrez et al., 1991)	LA (Auciello, 1993)
Pinned–pinned												
0.03	9.6949	9.695	9.748	9.676	10.1541	10.154	10.235	10.134	10.3774	10.377	10.458	10.36
0.04	9.5666	9.567	9.611	9.548	10.0071	10.007	10.045	9.988	10.2207	10.221	10.267	10.2
0.05	9.4106	9.411	9.446	9.393	9.8291	9.83	9.85	9.811	10.0314	10.031	10.054	10.01
0.06	9.2318	9.232	9.257	9.216	9.6265	9.627	9.645	9.61	9.8164	9.816	9.84	9.799
0.07	9.0356	9.036	9.057	9.021	9.4053	9.406	9.428	9.39	9.5825	9.582	9.604	9.566
0.08	8.8266	8.827	8.85	8.813	9.1713	9.172	9.189	9.157	9.3358	9.336	9.346	9.32

by Schmidt (1982). These results are different from all the others reported because of the numerical instabilities in the procedure. In the last column results are reported the results taken from a work of Auciello (1993) where the LA has been used. Unlike the other classical methods, the latter provides the convergence from below.

For the sake of completeness the remaining parameters Ω_2 ed Ω_3 are reported in Table 5.

In Tables 6 and 7 the results with boundary conditions $K_{T1} \to \infty$, $K_{T2} \to \infty$ and $K_{R1} = 0.1,\ K_{R2} = 0$ are reported. As can be easily verified the results are perfectly in agreement with the others.

5.3. The stability problem

The stability problem is obviously reduced to the following eigenvalue problem:

$$(\mathbf{K}_U - P_r\mathbf{K}_P)\mathbf{q} = 0. \tag{52}$$

The smallest P_r, solution of (52), gives the critical value while the related eigenvector produces the first order configuration. For $\alpha = 0$, the constant section beam the exact solution is given by the formula

Table 5
As a Table 4 for Ω_2 and Ω_3

α	0				0.1				0.15			
	Ω_2		Ω_3		Ω_2		Ω_3		Ω_2		Ω_3	
r	Present	LA (Auciello, 1993)	Present	LA (Auciello, 1993)	Present	LA (Auciello, 1993)	Present	LA (Auciello, 1993)	Present	LA (Auciello, 1993)	Present	LA (Auciello, 1993)
Pinned–pinned												
0.03	36.9565	36.681	77.954	76.526	38.546	38.226	80.934	79.423	39.3182	39.037	82.411	80.815
0.04	35.332	35.099	71.86	70.742	36.7218	36.487	74.212	73.043	37.3917	31.158	75.3655	74.137
0.05	33.572	33.373	65.971	65.084	34.7681	34.573	67.817	66.894	35.34	35.147	68.713	67.747
0.06	31.784	31.621	60.562	59.838	32.805	32.643	62.016	61.265	33.29	33.13	62.716	61.932
0.07	30.037	29.899	55.714	55.104	30.907	30.77	56.868	56.237	31.3176	31.183	57.419	56.763
0.08	28.372	28.254	51.415	50.888	29.1136	28.996	52.341	51.795	29.4618	29.346	52.781	52.213

Table 6
As a Table 4 for elastic spring ($K_{R1} = 0.1$)-pinned beam

α	0				0.1				0.15			
r	Present	Exact	Ritz approach (Gutierrez et al., 1991)	LA Auciello, 1993)	Present	FEM	(Gutierrez et al., 1991)	LA (Auciello, 1993)	Present	FEM	Ritz approach (Gutierrez et al., 1991)	LA (Auciello, 1993)
Pinned–pinned, $K_{R1} = 0.1$												
0.03	9.787	9.789	9.824	9.769	10.243	10.244	10.325	10.223	10.465	10.682	10.756	10.444
0.04	9.657	9.657	9.703	9.639	10.093	10.094	10.129	10.074	10.301	10.512	10.548	10.285
0.05	9.497	9.498	9.534	9.481	9.911	9.912	9.932	9.893	10.112	10.307	10.336	10.093
0.06	9.314	9.315	9.342	9.299	9.705	9.705	9.723	9.687	9.892	10.074	10.106	9.874
0.07	9.114	9.114	9.138	9.099	9.479	9.479	9.502	9.463	9.654	9.822	9.836	9.638
0.08	8.901	8.901	8.926	8.887	9.241	9.241	9.256	9.226	9.403	9.558	9.565	9.386

Table 7
As a Table 5 for elastic spring ($K_{R1} = 0.1$)-pinned beam

α	0				0.1				0.15			
	Ω_2		Ω_3		Ω_2		Ω_3		Ω_2		Ω_3	
r	Present	LA (Auciello, 1993)	Present	LA (Auciello, 1993)	Present	LA (Auciello, 1993)	Present	LA	Present	LA (Auciello, 1993)	Present	LA (Auciello, 1993)
Pinned–pinned, $K_{R1} = 0.1$												
0.03	37.009	36.763	77.571	76.595	38.6	38.346	80.479	79.489	39.374	39.117	81.882	80.881
0.04	35.382	35.173	71.539	70.799	36.771	36.559	73.841	73.097	37.441	37.228	74.938	74.19
0.05	33.616	33.442	65.705	65.13	34.81	34.636	67.513	66.938	35.383	35.209	68.368	67.79
0.06	31.822	31.678	60.341	59.875	32.842	32.698	61.766	61.301	33.327	33.184	62.436	61.967
0.07	30.071	29.949	55.527	55.135	30.938	30.818	56.662	56.266	31.349	31.229	57.19	56.791
0.08	28.401	28.297	51.258	50.913	29.141	29.038	52.168	51.818	29.489	29.386	52.59	52.236

Table 8
Compressive buckling load for different boundary conditions

α	Clamped-free	Pinned–pinned	$K_{R1} = 0.1$-Pinned	Clamped–clamped
−0.4	1.4653	3.9245	4.0163	10.4961
−0.2	1.8839	5.6787	5.7838	14.2016
0	2.291[a]	7.5459[a]	7.6605	17.6896[a]
0.1	2.4913	8.5079	8.6261	19.3318
0.2	2.6898	9.4822	9.6032	20.9015
0.3	2.8867	10.4645	10.5876	22.3954
0.4	3.0823	11.4504	11.5752	23.7648
0.6	3.4691	13.418	13.5449	26.2751
0.8	3.8529	15.3584	15.4851	28.3972
1	4.2327	17.2468	17.3718	30.1126

[a] Bazant–Cedolin; Eq. (58).

$$P_c = \frac{\pi^2 EI}{\widetilde{L}^2}\left(1 + \frac{\pi^2 EI}{\widetilde{L}^2 kGA}\right)^{-1}, \tag{53}$$

which takes into account the shear deformation too. Obviously the length $\widetilde{L}$ depends on the boundary conditions; Bažant and Cedolin (1991). In Table 8, for $r = 0.1$ and for different values of α some critical loads are reported. In particular they are referred to the following boundary conditions: clamped–clamped, clamped–free and pinned–pinned. The numerical results for the uniform beam are reported in italics and coincide with the ones obtained by the formula in (53).

6. Conclusions

The method presented, based on the Hamilton principle, obtains the solution minimizing the functional of the problem. The approximation lies both in the choice of the test functions and in the iterative procedure for the search of the minimum. The test functions are orthonormal polynomials which satisfy the essential conditions only and are automatically generated for every boundary conditions by the program of symbolic calculus *Mathematica*. It must be stressed that the latter has shown to be a powerful tool for both analytical and numerical developments. Comparing the results with the ones in literature some reflections can be pointed out.

(1) Compared to FEM a reduced number of parameters can be used with a consequently computer time saving.
(2) The proposed procedure has shown to be effective, very easy to implement and, compared to the optimization method with exponential factor, more stable especially for the higher frequencies.
(3) Assuming different test functions for displacements and rotations the shear locking can be limited. To avoid completely the locking phenomena higher order methods must be used as the one in Reddy (1997).
(4) Differently from other exact methods, the dynamic behaviour of tapered beams with various boundary conditions can be analysed without defining the relative test functions.

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